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Gradual Change Estimation in AR(1) Models

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ABSTRACT

Gradual change-point is an important topic in statistical studies. However, due to the complexity of gradual change-points, they are more difficult than other types of change-points and have scarcely been discussed in the literature. Thus, we highlight that, at an unknown time, a continuous type of change in an autocorrelated coefficient exists. Using Bayesian and maximum likelihood estimation methods, we tested the existence of trends in an autoregressive coefficient with a linear change. Simulation results, which support the Bayesian estimation method and compare it with the maximum likelihood estimation method, are reported for illustration.

Key words : AR(1) model, Bayesian estimation, Gradual changes, Maximum likelihood estimation

1. Introduction

Csörgő and Horváth [1] studied limit theories of abrupt change-points, as did Chen and Gupta [2] and Liu and Ha [3]. In addition, Lee et al. [4] studied the change point problems of parameters using CUSUM statistics. However, few studies have been conducted on gradual change-point problems, because of their complexity. Hušková [5] proposed a least square estimation method to study asymptotic properties in location parameter models, and Gupta and Ramanayake [6] discussed gradual change-point problems of exponential distribution parameters with a linear trend, and also discussed statistical properties of test statistics based on a generalized likelihood ratio test. In addition, Wang [7] detected the gradual change point of normal distribution parameters and provided a mathematical definition of gradual change points. Furthermore, Liu [8] developed a nonparametric method based on area under curve (AUC) to discuss gradual change points.

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In this paper, we partly reference methods based on Liu [9]. Here we consider the gradual change-point problems for the autocorrelated coefficient ϕ_0 in AR(1) model

$$X_t = \phi_0 X_{t-1} + \epsilon_t, t = 1, \cdots, n, \tag{1}$$

where $|\phi_0| < 1$, ϵ_t is an independent identically distributed random Gaussian errors and it satisfies

$$E(\epsilon_t) = 0, Var(\epsilon_t) = \sigma^2.$$

Now we will derive some statistics to test if the ϕ_0 is subject to a linear trend change at an unknown period of time. The hypothesis can be described as follows

$$H_0: X_t = \phi_0 X_{t-1} + \epsilon_t, t = 1, \cdots, n,$$

$$H_1: \begin{cases} X_t = \phi_0 X_{t-1} + \epsilon_t, t = 2, \cdots, k, \\ X_t = (\phi_0 + \delta(t-k))X_{t-1} + \epsilon_t, t = k+1, \cdots, n, \end{cases}$$

where the assumption for ϵ_t is the same whether autoregressive coefficients of AR(1) models change, and $k \in (2, n-1)$ is an unknown time called a gradual change-point. If null hypothesis H_0 holds true, there is no change-point, but if the alternative hypothesis H_1 holds true, there is a gradual change-point in AR(1) models.

2. Bayesian Estimation of Parameters

In Section 1, we obtained the hypothesis problems as follows:

$$H_0: X_t = \phi_0 X_{t-1} + \epsilon_t, t = 1, \cdots, n,$$

$$H_1: \begin{cases} X_t = \phi_0 X_{t-1} + \epsilon_t, t = 2, \cdots, k, \\ X_t = (\phi_0 + \delta(t-k))X_{t-1} + \epsilon_t, t = k+1, \cdots, n. \end{cases}$$

For given X_t, we set $\boldsymbol{\theta} = (k, \phi_0, \delta, \sigma^2)^T$, the hypothesis problems then have a likelihood function as follows:

$$L = (2\pi\sigma^2)^{-\frac{n-1}{2}} \cdot exp\left\{-\frac{1}{2\sigma^2} \cdot [\sum_{t=2}^{k} (X_t - \phi_0 X_{t-1})^2 + \sum_{t=k+1}^{n} (X_t - (\phi_0 + \delta(t-k))X_{t-1})^2]\right\}$$

$$= (2\pi\sigma^2)^{-\frac{n-1}{2}} \cdot exp\left\{-\frac{1}{2\sigma^2} \cdot [\sum_{t=2}^{n} (X_t - \phi_0 X_{t-1})^2 - 2\delta \sum_{t=k+1}^{n} (t-k)(X_t - \phi_0 X_{t-1})X_{t-1}]\right\} \cdot \exp\left\{-\frac{1}{2\sigma^2} \cdot \delta^2 \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2\right\}$$

$$= (2\pi\sigma^2)^{-\frac{n-1}{2}} \cdot exp\left\{-\frac{1}{2\sigma^2} \cdot [\sum_{t=2}^{n} (X_t - \phi_0 X_{t-1})^2 - 2\delta \sum_{t=k+1}^{n} (t-k)X_t X_{t-1} + 2\delta\phi_0 \sum_{t=k+1}^{n} (t-k)X_{t-1}^2]\right\} \cdot \exp\left\{-\frac{1}{2\sigma^2} \cdot \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2\right\}.$$

We assume the prior distributions of ϕ_0, δ, σ^2 are all uniform distribution and we take the prior distribution $\pi(k)$ of k from Yang [10], viz., $\pi(k) \propto k(n-k)$, $k = 2, \cdots, n$. Additionally, prior distributions of all parameters are independent of each other. The notations that appear in the text and the appendix are the same unless otherwise noted.
Let

$$M_1 = \sum_{t=2}^{n} (X_t - \phi_0 X_{t-1})^2,$$

$$M_2 = -2\delta \sum_{t=k+1}^{n} (t-k)(X_t - \phi_0 X_{t-1})X_{t-1},$$

$$M_3 = -2\delta \sum_{t=k+1}^{n} (t-k)X_t X_{t-1},$$

$$M_4 = 2\delta\phi_0 \sum_{t=k+1}^{n} (t-k)X_{t-1}^2, M_5 = \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2,$$

then $M_2 = M_3 + M_4$.

As proposition of Bayes distribution, the joint posterior distribution of all parameters is given by

$$\pi(\boldsymbol{\theta} \mid X_t) \propto L \cdot \pi(k)/\sigma^2$$

$$= (2\pi\sigma^2)^{-\frac{n-1}{2}} \cdot exp\left\{-\frac{1}{2\sigma^2} \cdot [M_1 + M_3 + M_4 + M_5]\right\} \cdot \frac{\pi(k)}{\sigma^2}.$$

Then the kernel of change point k is

$$\pi(k \mid x_t) \propto \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} (2\pi\sigma^2)^{-\frac{n-1}{2}} \cdot exp\left\{-\frac{1}{2\sigma^2} \cdot [M_1 + M_3 + M_4 + M_5]\right\} \cdot \frac{\pi(k)}{\sigma^2} d\sigma^2 d\phi_0 d\delta$$

$$\propto (a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(k),$$

where

$$a_1 = \sum_{t=2}^{n} X_{t-1}^2, a_2 = \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2,$$

$$b_2 = \sum_{t=k+1}^{n} (t-k)X_t X_{t-1}, c_2 = \sum_{t=k+1}^{n} (t-k)X_{t-1}^2,$$

$$d_2 = \sum_{t=2}^{n} X_t X_{t-1}, e_2 = \sum_{t=2}^{n} X_t^2.$$

As k is discrete, its posterior distribution is

$$\pi(k \mid x_t) = \frac{(a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(k)}{\sum_{l=2}^{n} (a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(l)}, \quad 2 \le k \le n.$$

Then the Bayesian estimator $\hat{k}$ of change-point k is $\hat{k} = \underset{2 \le k \le n}{argmax} \pi(k \mid x_t)$.

Similarly, as properties of t-distribution and Γ-distribution, we can obtain the kernels of conditional posterior distributions of parameters σ^2, δ and ϕ_0 respectively:

$$\pi(\sigma^2 \mid x_t) \propto \sum_{k=2}^{n-1} (\sigma^2)^{-\frac{n-1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}} \cdot \exp\left\{-\frac{e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}}{2\sigma^2}\right\} \pi(k \mid x_t),$$

$$\pi(\delta \mid x_t) \propto \sum_{k=2}^{n-1} (c_1 + M_3 + M_5)^{-\frac{n-2}{2}} (a_2 - \frac{c_2^2}{a_1})^{\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}} \cdot \pi(k \mid x_t),$$

where

$$c_1 = -\frac{\left[\delta \sum_{t=k+1}^{n} (t-k)X_{t-1}^2 - \sum_{t=2}^{n} X_t X_{t-1}\right]^2}{\sum_{t=2}^{n} X_{t-1}^2} + \sum_{t=2}^{n} X_t^2$$

$$= -a_1 \cdot b_1^2 + \sum_{t=2}^{n} X_t^2,$$

$$\pi(\phi_0 \mid x_t) \propto \sum_{k=2}^{n-1} (M_1 - \frac{(b_2 - \phi_0 c_2)^2}{a_2})^{-\frac{n-2}{2}} (a_1 a_2 - c_2^2)^{\frac{1}{2}}$$

$$\cdot \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}} \right]^{\frac{n-3}{2}} a_2^{-\frac{1}{2}} \pi(k \mid x_t).$$

Correspondingly, the posterior distributions of parameters are:

$$\pi(\sigma^2 \mid x_t) = \sum_{k=2}^{n-1} IGa(\sigma^2; \frac{n-3}{2}, \frac{e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}}{2}) \cdot \pi(k \mid x_t),$$

$$\pi(\delta \mid x_t) =$$

$$\sum_{k=2}^{n-1} t\left(\delta; n-3, -\frac{\frac{c_2 d_2}{a_1} - b_2}{a_2 - \frac{c_2^2}{a_1}}, \sqrt{(e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}})/(n-3)(a_2 - \frac{c_2^2}{a_1})}\right)$$

$$\cdot \left(e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right)^{n-3} \pi(k \mid x_t),$$

$$\pi(\phi_0 \mid x_t) = \sum_{k=2}^{n-1} t\left(\phi_0; n-3, -\frac{b_2 c_2 - a_2 d_2}{a_1 a_2 - c_2^2}, \sqrt{\frac{a_2 e_2 - b_2^2 - \frac{(b_2 c_2 - a_2 d_2)^2}{a_1 a_2 - c_2^2}}{(n-3)(a_1 a_2 - c_2^2)}}\right) \cdot a_2^{\frac{n-3}{2}}$$

$$\cdot \left(e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right)^{\frac{n-3}{2}} (a_2 e_2 - b_2^2 - \frac{(b_2 c_2 - a_2 d_2)^2}{a_1 a_2 - c_2^2})^{\frac{n-3}{2}} \pi(k \mid x_t).$$

Proof: See A.1.2., A.1.3., and A.1.4. in the Appendix.

Therefore, the Bayesian estimators are:

$$\hat{\sigma}^2 = E(\sigma^2 \mid x_t) = \sum_{k=2}^{n-1} \frac{e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}}{n-5} \pi(k \mid x_t),$$

$$\hat{\delta} = \sum_{k=2}^{n-1} \frac{\frac{b_2 - c_2 d_2}{a_1}}{a_2 - \frac{c_2^2}{a_1}} \left(e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right)^{n-3} \pi(k \mid x_t),$$

$$\hat{\phi}_0 = \sum_{k=2}^{n-1} \frac{a_2 d_2 - b_2 c_2}{a_1 a_2 - c_2^2} \left(e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right)^{\frac{n-3}{2}}$$

$$\cdot (a_2 e_2 - b_2^2 - \frac{(b_2 c_2 - a_2 d_2)^2}{a_1 a_2 - c_2^2})^{\frac{n-3}{2}} a_2^{\frac{n-3}{2}} \pi(k \mid x_t).$$

3. Maximum Likelihood Estimation

For detecting gradual change point by using the maximum likelihood method, Wang [7] discussed the topic using two situations: when the autocorrelated coefficient ϕ_0 is known and unknown in the first-order autoregressive time series models.

(1) The variance σ^2 is known and the autocorrelated coefficient ϕ_0 has a known initial value;

(2) The variance σ^2 is known but the autocorrelated coefficient ϕ_0 is unknown.

In situation (1), for fixed k, the maximum likelihood estimate of parameter δ under the alternative hypothesis H_1 is

$$\hat{\delta} = \sum_{t=k+1}^{n} (t-k)(x_t - \phi_0 x_{t-1}) x_{t-1} / \sum_{t=k+1}^{n} (t-k)^2 x_{t-1}^2,$$

then the test statistic for testing the change point k is given by

$$\Lambda_n^* = \max_{2 \le k \le n} |\Lambda_k|,$$

where

$$\Lambda_k = \sum_{t=k+1}^{n} ((t-k)(x_t - \phi_0 x_{t-1}) x_{t-1})^2 / (\sum_{t=k+1}^{n} \sigma^2 (t-k)^2 x_{t-1}^2).$$

In another situation, the variance σ^2 is known but the autocorrelated coefficient ϕ_0 is unknown, Wang [7] gave the estimates of parameters ϕ_0 and δ as follows

$$\hat{\phi}_0 = \sum_{t=2}^{n} x_t x_{t-1} / \sum_{t=2}^{n} x_{t-1}^2$$

and

$$\hat{\delta} = \sum_{t=k+1}^{n} (t-k)(x_t - \hat{\phi}_0 x_{t-1}) x_{t-1} / \sum_{t=k+1}^{n} (t-k)^2 x_{t-1}^2.$$

Thus, the test statistic detecting the gradual change point for the autocorrelated coefficient ϕ_0 can be obtained as:

$$G_n^* = \max_{2 \le k \le n} |G_k|$$

where

$$G_k = \sum_{t=k+1}^{n} ((t-k)(x_t - \hat{\phi}_0 x_{t-1}) x_{t-1})^2 / (\sum_{t=k+1}^{n} \sigma^2 (t-k)^2 x_{t-1}^2).$$

Since it is hard to obtain its limit distribution, so we propose using the stochastic simulation method to obtain asymptotic critical values.

4. Simulation

In this section, we evaluate whether the techniques are effective and feasible or not through a simulation study. We obtained the Bayesian estimators of all parameters and maximum likelihood estimators in Section 2 and Section 3 respectively. Firstly, sets of 100 observations are generated from the stationary AR(1) model with $\phi_0 = -0.01$, $\delta = 0.03$ and $k = 70$, where ϵ_t are normal random variables with a mean of zero and $\sigma^2 = 0.01$. Then the corresponding time series plot is shown in Fig. 1. By the way, Fig. 2 shows the function curve of the posterior distribution $\pi(k|x_t)$ for change-point $k = 2,3,\cdots,99$. And the largest point is just 70 which is the change-point k.

In order to further assess the reliability of the method, sets of 200 observations are produced from another stationary AR(1) model with $\phi_0 = 0.8$, $\delta = -0.01$ and $k = 100$, where ϵ_t are also normal random variables with a mean of zero and $\sigma^2 = 0.01$. Then the corresponding time plot is given by Fig. 3. Correspondingly, the function curve of the posterior distribution $\pi(k|x_t)$ is shown by Fig. 4. Then the maximum point of the calculated final largest $\pi(k|x_t)$ is $k = 106$, which is close to the predetermined change-point $k = 100$. The time difference is $\Delta(t) = 6$, so the relative error $\Delta(t)/T(n) = 3.0\%$.

Next, to determine the reliability of the maximum likelihood approach, we will simulate in the same way. But the time plots will be omitted for simplicity. The predetermined parameters will be the same. Thus, sets of 500 observations generated from a stationary AR(1) model with $\phi_0 = 0.6$, $\delta = -0.01$ and $k = 400$, where $\epsilon_t \sim N(0,0.01)$. Under the null hypothesis H_0, when $k = 2,3,\cdots,500$, the function graph of the test statistic Λ_n^* is shown by Fig. 5. Then the maximum point of the calculated final largest Λ_n^* is $t = 409$ and the time difference is $\Delta(t) = 9$, so the relative error is 1.8%.

For the test statistic G_n^*, its function curve is shown by Fig. 6. And then the calculated maximum point is $t = 385$, so the

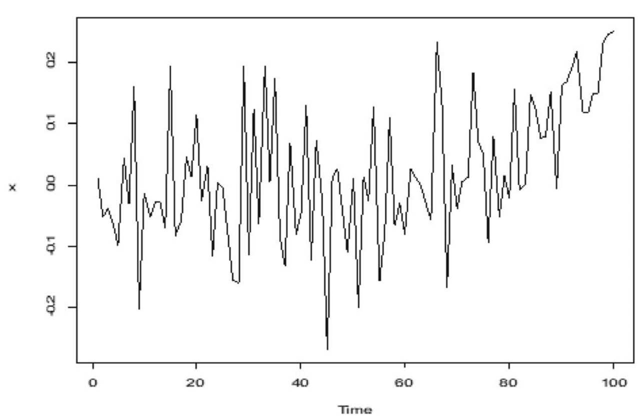

Fig. 1. Time plot with $\phi_0 = -0.01$ and $\delta = 0.03$ (Bayes method).

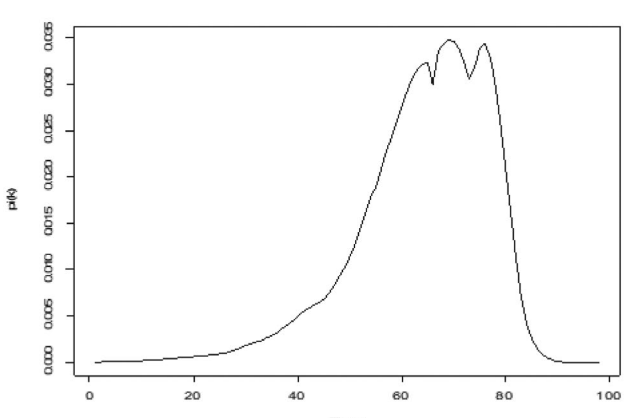

Fig. 2. $\pi(k|x_t)$ curve with $\phi_0 = -0.01$ and $\delta = 0.03$.

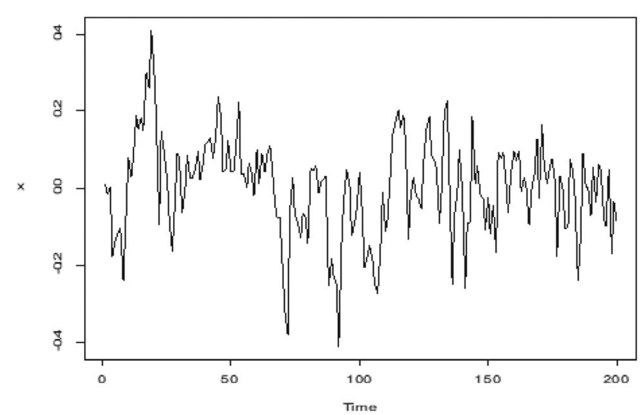

Fig. 3. Time plot with $\phi_0 = 0.8$ and $\delta = -0.01$ (Bayes method).

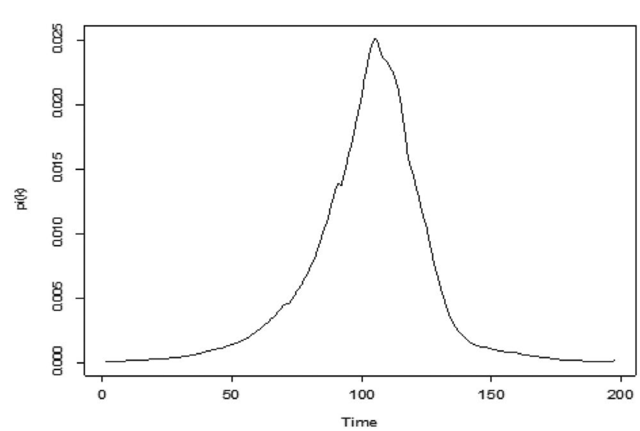

Fig. 4. $\pi(k|x_t)$ curve with $\phi_0 = 0.8$ and $\delta = -0.01$.

relative error is 3.0%. From the above results, we see that both the Bayes and maximum likelihood methods have same good performance for change-point estimation. Additionally, the final relative error results are less than 5%, so two techniques are reliable and feasible.

In addition, to test the power of the maximum likelihood strategy, we take a new simulation study into account. We deal with Λ_n^* and G_n^* in Section 3. For the empirical sizes of Λ_n^* and G_n^* sets of 100, 300, and 500 observations are generated from the AR(1) model with $\phi_0 = 0.1, 0.3, 0.5$, and $\delta = -0.001$, and the level of the test is $\alpha = 0.05$, $\epsilon_t \sim N(0, 0.01)$. Five hundred simulations are simultaneously implemented. We consider the situations as follows:

(1) The variance σ^2 is known and the autocorrelated coefficient ϕ_0 has a known initial value;

(2) The variance σ^2 is known but the autocorrelated coefficient ϕ_0 is unknown.

Table 1 and Table 2 show the empirical sizes and powers of Λ_n^* and G_n^* when level is $\alpha = 0.05$, $\sigma^2 = 0.01$ and $\delta = -0.001$ respectively. Although the empirical sizes and powers do not differ greatly from each other because of the sample size, it is clear that the powers increase with sample size n. In addition, the above outcomes on powers indicate that the maximum likelihood method has good detection performance for gradual change points; thus, it is worth rationalizing and validating the theory further.

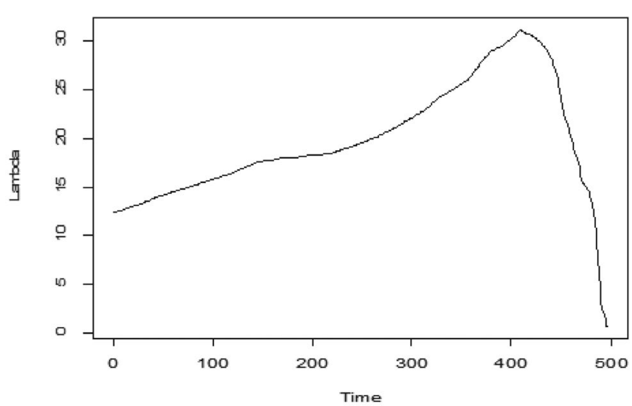

Fig. 5. Test statistic Λ_n^* curve.

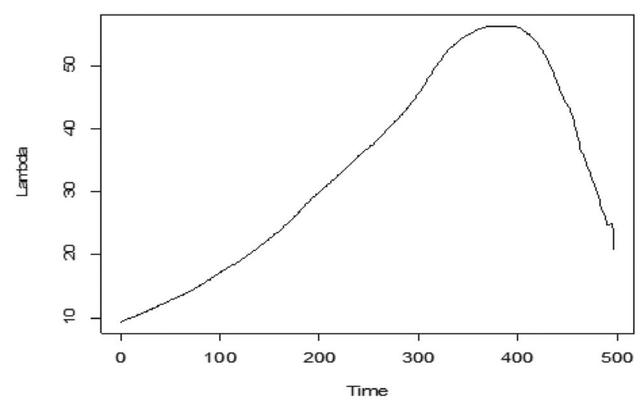

Fig. 6. Test statistic G_n^* curve.

Table 1. Empirical sizes and powers of Λ_n^* ($\alpha = 0.05$, $\delta = -0.001$ and $\sigma^2 = 0.01$)

ϕ_0		Size			Power		
		0.1	0.3	0.5	0.1	0.3	0.5
n	100	0.0460	0.0480	0.0540	0.9440	0.9540	0.9200
	300	0.0300	0.0500	0.0620	0.9660	0.9620	0.9460
	500	0.0360	0.0340	0.0340	0.9720	0.9680	0.9500

Table 2. Empirical sizes and powers of G_n^* ($\alpha = 0.05$, $\delta = -0.001$ and $\sigma^2 = 0.01$)

ϕ_0		Size			Power		
		0.1	0.3	0.5	0.1	0.3	0.5
n	100	0.0620	0.0480	0.0380	0.9320	0.9400	0.9520
	300	0.0400	0.0440	0.0320	0.9420	0.9620	0.9600
	500	0.0220	0.0300	0.0340	0.9780	0.9660	0.9620

5. Conclusion

This article discusses the fact that at an unknown time, a continued type of change exists in the autoregressive coefficient such that the trend in that coefficient after the change is linear. It does so by using the Bayesian estimation and maximum likelihood estimation methods when the variance is the same before and after the change point, respectively. In Section 2, we not only provided the posterior distributions of all parameters, but also reveal their Bayesian estimation values. In Section 3, we also discuss maximum likelihood estimators of gradual change points, as well as some parameters. In sum, the findings (shown in above figures) clearly illustrate that the Bayesian approach is equally applicable as the maximum likelihood approach for determining the position of gradual change point. Moreover, the maximum likelihood method has a high power for testing change points, as seen in Tables 1 and 2, and the feasibility and reliability of our strategies in this paper can also be verified through simulation studies.

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Appendix : Proofs

A.1. Proof of kernels of all parameters

Here, we provide proofs of the kernels of parameters k, σ^2, δ and ϕ_0 in four parts.

A.1.1. Proof of kernel of *k*

Here we let

$$I_1 = \int_0^{+\infty} (2\pi)^{-\frac{n-1}{2}} \cdot \frac{1}{(\sigma^2)^{\frac{n-1}{2}+1}} \cdot \exp\left\{-\frac{M_1+M_3+M_n+M_5}{2\cdot\sigma^2}\right\} d\sigma^2$$

$$\propto \int_0^{+\infty} \frac{1}{(\sigma^2)^{\frac{n-1}{2}+1}} \cdot \exp\left\{-\frac{M_1+M_3+M_n+M_5}{2\sigma^2}\right\} d\sigma^2$$

$$= \Gamma(\frac{n-1}{2})(-\frac{M_1+M_3+M_n+M_5}{2})^{-\frac{n-1}{2}} \int_0^{+\infty} \frac{(\frac{M_1+M_3+M_n+M_5}{2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})(\sigma^2)^{\frac{n-1}{2}+1}} \exp(-\frac{\frac{M_1+M_3+M_4+M_5}{2}}{\sigma^2}) d\sigma^2$$

$$= 2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})(M_1+M_3+M_4+M_5)^{-\frac{n-1}{2}}$$

then let

$$I_2 = \int_{-\infty}^{+\infty} 2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})(M_1+M_3+M_4+M_5)^{-\frac{n-1}{2}} d\phi_0$$

$$\propto \int_{-\infty}^{+\infty} (M_1+M_3+M_4+M_5)^{-\frac{n-1}{2}} d\phi_0$$

where

$$M_1 = \sum_{t=2}^n (X_t - \phi_0 X_{t-1})^2 = \phi_0^2 \sum_{t=2}^n X_{t-1}^2 - 2\phi_0 \sum_{t=2}^n X_t X_{t-1} + \sum_{t=2}^n X_t^2, M_4 = 2\delta\phi_0 \sum_{t=k+1}^n (t-k) X_{t-1}^2,$$

such that

$$M_1 + M_4 = \phi_0^2 \sum_{t=2}^n X_{t-1}^2 - 2\phi_0 \sum_{t=2}^n X_t X_{t-1} + \sum_{t=2}^n X_t^2 + 2\delta\phi_0 \sum_{t=k+1}^n (t-k) X_{t-1}^2$$

$$= \phi_0^2 \sum_{t=2}^n X_{t-1}^2 + 2\phi_0[\delta \sum_{t=k+1}^n (t-k)X_t^2 - \sum_{t=2}^n X_t X_{t-1}] + \sum_{t=2}^n X_t^2$$

$$= \sum_{t=2}^n X_{t-1}^2 \cdot \left[\phi_0^2 + 2\phi_0 \frac{\delta \sum_{t=k+1}^n (t-k)X_{t-1}^2 - \sum_{t=2}^n X_t X_{t-1}}{\sum_{t=2}^n X_{t-1}^2}\right] + \sum_{t=2}^n X_t^2$$

$$= \sum_{t=2}^n X_{t-1}^2 \left[\phi_0 + \frac{\delta \sum_{t=k+1}^n (t-k)X_{t-1}^2 - \sum_{t=2}^n X_t X_{t-1}}{\sum_{t=2}^n X_{t-1}^2}\right]^2 - \frac{\left[\delta \sum_{t=k+1}^n (t-k)X_{t-1}^2 - \sum_{t=2}^n X_t X_{t-1}\right]^2}{\sum_{t=2}^n X_{t-1}^2} + \sum_{t=2}^n X_t^2.$$

Let

$$a_1 = \sum_{t=2}^n X_{t-1}^2, b_1 = \frac{\delta \sum_{t=k+1}^n (t-k)X_{t-1}^2 - \sum_{t=2}^n X_t X_{t-1}}{\sum_{t=2}^n X_{t-1}^2},$$

$$c_1 = -\frac{\left[\delta \sum_{t=k+1}^n (t-k)X_{t-1}^2 - \sum_{t=2}^n X_t X_{t-1}\right]^2}{\sum_{t=2}^n X_{t-1}^2} + \sum_{t=2}^n X_t^2 = -a_1 \cdot b_1^2 + \sum_{t=2}^n X_t^2,$$

then

$$M_1 + M_4 = a_1 \cdot (\phi_0 + b_1)^2 + c_1.$$

As *t*-distribution, we can obtain

$$I_2 \propto \int_{-\infty}^{+\infty} [a_1 \cdot (\phi_0 + b_1)^2 + c_1 + M_3 + M_5]^{-\frac{n-1}{2}} d\phi_0$$

$$= \int_{-\infty}^{+\infty} (c_1 + M_3 + M_5)^{-\frac{n-1}{2}} \cdot \left[1 + \frac{a_1(\phi_0+b_1)^2}{c_1+M3+M_5}\right]^{-\frac{n-1}{2}} d\phi_0$$

$$= \frac{(c_1+M_3+M_5)^{-\frac{n-1}{2}}\Gamma(\frac{n-2}{2})\sqrt{(n-2)\pi}}{\sqrt{a_1(n-2)/(c_1+M_3+M_5)}\cdot\Gamma(\frac{n-1}{2})}$$

$$\cdot \int_{-\infty}^{+\infty} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})\sqrt{(n-2)\pi}} \left[1 + \frac{\left[\sqrt{a_1(n-2)/(c_1+M_3+M_5)}(\phi_0+b_1)\right]^2}{n-2}\right] d\sqrt{a_1(n-2)/(c_1+M_3+M_5)}(\phi_0+b_1)$$

$$\propto (c_1+M_3+M_5)^{-\frac{n-2}{2}} \cdot a_1^{-\frac{1}{2}}.$$

Now we consider the integration of δ, due to

$$c_1+M_3+M_5 = \delta^2 \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2 - 2\delta \sum_{t=k+1}^{n} (t-k) X_t X_{t-1} - \frac{\left[\delta \sum_{t=k+1}^{n} (t-k) X_{t-1}^2 - \sum_{t=2}^{n} X_t X_{t-1}\right]^2}{\sum_{t=2}^{n} X_{t-1}^2} + \sum_{t=2}^{n} X_t^2.$$

Similarly, we can also let

$$a_2 = \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2, b_2 = \sum_{t=k+1}^{n} (t-k) X_t X_{t-1}, c_2 = \sum_{t=k+1}^{n} (t-k) X_{t-1}^2, d_2 = \sum_{t=2}^{n} X_t X_{t-1}, e_2 = \sum_{t=2}^{n} X_t^2,$$

then

$$\begin{aligned} c_1+M_3+M_5 &= \delta^2 \cdot a_2 - 2\delta \cdot b_2 - \frac{1}{a_1}(\delta^2 \cdot c_2^2 - 2c_2 d_2 \delta + d_2^2) + e_2 \\ &= \delta^2 \cdot a_2 - 2\delta \cdot b_2 - \frac{c_2^2}{a_1}\delta^2 + \frac{c_2 d_2}{a_1} \cdot 2\delta - \frac{d_2^2}{a_1} + e_2 \\ &= (a_2 - \frac{c_2^2}{a_1})\delta^2 + 2(\frac{c_2 d_2}{a_1} - b_2)\delta + e_2 - \frac{d_2^2}{a_1} \\ &= (a_2 - \frac{c_2^2}{a_1}) \left[\delta + \frac{\frac{c_2 d_2}{a_1} - b_2}{a_2 - \frac{c_2^2}{a_1}}\right]^2 + e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}. \end{aligned}$$

Hence

$$\begin{aligned} I_3 &= \int_{-\infty}^{+\infty} (c_1+M_3+M_5)^{-\frac{n-2}{2}} a_1^{-\frac{1}{2}} d\delta \\ &= a_1^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-2}{2}} \cdot \left[1 + \frac{(a_2 - \frac{c_2^2}{a_1})\left[\delta + \frac{\frac{c_2 d_2}{a_1} - b_2}{a_2 - \frac{c_2^2}{a_1}}\right]^2}{e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}}\right]^{-\frac{n-2}{2}} d\delta \\ &\propto a_1^{-\frac{1}{2}} (a_2 - \frac{c_2^2}{a_1})^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \\ &= (a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}}. \end{aligned}$$

Therefore, the kernel of posterior distribution of k can be given by

$$\pi(k \mid x_t) \propto (a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(k).$$

A.1.2. Proof of the kernel σ^2

We consider the kernel of posterior distribution of σ^2

$$\pi(\sigma^2 \mid x_t) \propto \sum_{k=2}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (2\pi\sigma^2)^{-\frac{n-1}{2}} \cdot exp\left\{-\frac{1}{2\sigma^2} \cdot [M_1+M_3+M_4+M_5]\right\} \cdot \frac{\pi(k)}{\sigma^2} d\phi_0 d\delta.$$

Let

$$P_1 = \int_{-\infty}^{+\infty} \exp\left\{-\frac{M_1+M_4}{2\sigma^2}\right\} d\phi_0 = \int_{-\infty}^{+\infty} \exp\left\{-\frac{a_1\cdot(\phi_0+b_1)^2+c_1}{2\sigma^2}\right\} d\phi_0$$

and set

$$x = \sqrt{a_1}(\phi_0 + b_1)$$

then

$$P_1 = \int_{-\infty}^{+\infty} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \exp\left\{-\frac{c_1}{2\sigma^2}\right\} \frac{1}{\sqrt{a_1}} dx.$$

As the properties of χ^2 distribution

$$P_1 = \exp\left\{-\frac{c_1}{2\sigma^2}\right\} \frac{1}{\sqrt{a_1}} \sqrt{2\pi}\sigma \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = \exp\left\{-\frac{c_1}{2\sigma^2}\right\} \frac{1}{\sqrt{a_1}} \sqrt{2\pi}\sigma.$$

Similarly, let

$$P_2 = \int_{-\infty}^{+\infty} \exp\left\{-\frac{c_1+M_3+M_5}{2\sigma^2}\right\} d\delta \; = \exp\left\{-\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2\sigma^2}\right\} \frac{1}{\sqrt{a_2-\frac{c_2^2}{a_1}}} \sqrt{2\pi}\sigma.$$

So

$$\pi(\sigma^2 \mid x_t) \propto \sum_{k=2}^{n-1} (2\pi\sigma^2)^{-\frac{n-1}{2}} \frac{\pi(k)}{\sigma^2} a_1^{-\frac{1}{2}} (a_2 - \frac{c_2^2}{a_1})^{-\frac{1}{2}} (2\pi\sigma^2) \exp\left\{-\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2\sigma^2}\right\}$$

$$= \sum_{k=2}^{n-1} (2\pi\sigma^2)^{-\frac{n-1}{2}} \frac{\pi(k)}{\sigma^2} (a_1 a_2 - c_2^2)^{-\frac{1}{2}} (2\pi\sigma^2) \exp\left\{-\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2\sigma^2}\right\}$$

$$\propto \sum_{k=2}^{n-1} (\sigma^2)^{-\frac{n-1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(k) \cdot (a_1 a_2 - c_2^2)^{-\frac{1}{2}}$$

$$\cdot \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}} \exp\left\{-\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2\sigma^2}\right\}$$

$$\propto \sum_{k=2}^{n-1} (\sigma^2)^{-\frac{n-1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}} \exp\left\{-\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2 d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2\sigma^2}\right\} \pi(k \mid x_t).$$

A.1.3. Proof of the kernel δ

We derive the posterior distribution of δ according to the properties of t-distribution and inverse Γ-distribution, we then the posterior distribution kernel of δ, which can be given by

$$\pi(\delta \mid x_t) \propto \sum_{k=2}^{n-1} \pi(k) \int_{-\infty}^{+\infty} \int_0^{+\infty} (2\pi\sigma^2)^{-\frac{n-1}{2}} exp\left\{-\frac{M_1+M_3+M_4+M_5}{2\sigma^2}\right\} \cdot \frac{1}{\sigma^2} d\sigma^2 d\phi_0$$

$$\propto \sum_{k=2}^{n-1} \pi(k) \int_{-\infty}^{+\infty} (M_1+M_3+M_4+M_5)^{-\frac{n-1}{2}} \int_0^{+\infty} \frac{(\frac{M_1+M_3+M_4+M_5}{2})^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})(\sigma^2)^{\frac{n-1}{2}+1}} exp\left\{-\frac{M_1+M_3+M_4+M_5}{2\sigma^2}\right\} d\sigma^2 d\phi_0$$

$$\propto \sum_{k=2}^{n-1} \pi(k) \int_{-\infty}^{+\infty} (M_1+M_3+M_4+M_5)^{-\frac{n-1}{2}} d\phi_0.$$

The kenel of δ can be obtained as:

$$\pi(\delta \mid x_t) \propto \sum_{k=2}^{n-1} \pi(k)(c_1+M_3+M_5)^{-\frac{n-1}{2}} a_1^{-\frac{1}{2}}$$

$$\propto \sum_{k=2}^{n-1} (c_1+M_3+M_5)^{-\frac{n-2}{2}} (a_2 - \frac{c_2^2}{a_1})^{\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}} \pi(k \mid x_t).$$

A.1.4. Proof of the kernel ϕ_0

Similarly, we can also obtain the posterior distribution kernel of ϕ_0

$$\pi(\phi_0 \mid x_t) \propto \sum_{k=2}^{n-1} \pi(k) \int_{-\infty}^{+\infty} \int_0^{+\infty} (2\pi\sigma^2)^{-\frac{n-1}{2}} exp\left\{-\frac{M_1+M_3+M_4+M_5}{2\sigma^2}\right\} \cdot \frac{1}{\sigma^2} d\sigma^2 d\delta$$

$$\propto \sum_{k=2}^{n-1} \pi(k) \int_{-\infty}^{+\infty} (M_1+M_2+M_5)^{-\frac{n-1}{2}} d\delta.$$

where

$$M_2 + M_5 = \delta^2 \sum_{t=k+1}^{n} (t-k)^2 X_{t-1}^2 - 2\delta \sum_{t=k+1}^{n} (t-k)(X_t - \phi_0 X_{t-1}) X_{t-1} = M_1 - \frac{(b_2 - \phi_0 c_2)^2}{a_2},$$

then

$$\pi(\phi_0 \mid x_t) \propto \sum_{k=2}^{n-1} \pi(k) \cdot (M_1 - \frac{(b_2-\phi_0 c_2)^2}{a_2})^{-\frac{n-2}{2}} a_2^{-\frac{1}{2}}$$

$$\propto \sum_{k=2}^{n-1} (M_1 - \frac{(b_2-\phi_0 c_2)^2}{a_2})^{-\frac{n-2}{2}} (a_1 a_2 - c_2^2)^{\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}} a_2^{-\frac{1}{2}} \pi(k \mid x_t).$$

A.2. Proof of Bayesian estimators of parameters

We also divide this part into four parts to obtain the Bayesian estimators of parameters k, σ^2, δ and ϕ_0.

A.2.1. Proof of Bayesian estimator of k

We have obtained the kernel of posterior distribution of change-point k

$$\pi(k \mid x_t) \propto (a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(k).$$

As k is discrete, its posterior distribution is

$$\pi(k \mid x_t) = \frac{(a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(k)}{\sum_{l=2}^{n} (a_1 a_2 - c_2^2)^{-\frac{1}{2}} \left[e_2 - \frac{d_2^2}{a_1} - \frac{(\frac{c_2 d_2}{a_1} - b_2)^2}{a_2 - \frac{c_2^2}{a_1}}\right]^{-\frac{n-3}{2}} \pi(l)}, \quad 2 \le k \le n,$$

then the Bayesian estimator $\hat{k}$ of change-point k is $\hat{k} = \underset{2 \le k \le n}{argmax} \pi(k \mid x_t)$.

A.2.2. Proof of Bayesian estimator of σ^2

As a *t*-distribution, its density function is

$$f(x;n,\mu,\sigma^2)=\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}\sigma}\cdot\left[1+\frac{(x-\mu)^2}{n\sigma^2}\right]^{-\frac{n+1}{2}}$$

where n is degree of freedom, μ is location parameter and σ is scale parameter, then $E(f)=\mu$. In addition, for inverse gamma distribution, written as $IGa(\alpha,\beta)$, its density function is

$$IGa(x;\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)x^{\alpha+1}}\cdot e^{-\frac{\beta}{\alpha}},x>0$$

where both α and β are parameters, then $E(IGa)=\frac{\beta}{\alpha-1}$. Finally, for $\chi^2(n)$ with degree of freedom n, its density function is

$$f_{\chi^2}(x;n)=\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\cdot x^{\frac{n}{2}-1}\cdot e^{-\frac{x}{2}},x>0.$$

Hence,

$$\pi(\sigma^2\mid x_t)\propto\sum_{k=2}^{n-1}(\sigma^2)^{-\frac{n-1}{2}}\left[e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}}\exp\left\{-\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2\sigma^2}\right\}\pi(k\mid x_t)$$

$$=\sum_{k=2}^{n-1}\frac{\left(\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2}\right)^{\frac{n-2}{2}}}{\Gamma(\frac{n-3}{2})(\sigma^2)^{-\frac{n-1}{2}}}\exp\left\{-\frac{\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2}}{\sigma^2}\right\}\pi(k\mid x_t)$$

$$=\sum_{k=2}^{n-1}IGa\left(\sigma^2;\frac{n-3}{2},\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{2}\right)\pi(k\mid x_t).$$

The Bayesian estimator of σ^2 is

$$\hat{\sigma}^2=E(\sigma^2\mid x_t)=\sum_{k=2}^{n-1}\frac{e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}}{n-5}\pi(k\mid x_t).$$

A.2.3. Proof of Bayesian estimator of δ

$$\pi(\delta\mid x_t)\propto\sum_{k=2}^{n-1}(c_1+M_3+M_5)^{-\frac{n-2}{2}}(a_2-\frac{c_2^2}{a_1})^{\frac{1}{2}}\left[e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}}\pi(k\mid x_t)$$

$$=\sum_{k=2}^{n-1}\left[(a_2-\frac{c_2^2}{a_1})\left(\delta+\frac{\frac{c_2d_2}{a_1}-b_2}{a_2-\frac{c_2^2}{a_1}}\right)^2+e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{-\frac{n-2}{2}}(a_2-\frac{c_2^2}{a_1})^{\frac{1}{2}}\cdot\left[e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}}\pi(k\mid x_t)$$

$$=\sum_{k=2}^{n-1}(a_2-\frac{c_2^2}{a_1})^{\frac{1}{2}}\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{n-3}\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{-\frac{1}{2}}$$

$$\cdot\left[1+\frac{\sqrt{(n-3)(a_2-\frac{c_2^2}{a_1})/(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}})}\left(\delta+\frac{\frac{c_2d_2}{a_1}-b_2}{a_2-\frac{c_2^2}{a_1}}\right)^2}{n-3}\right]^{-\frac{n-3+1}{2}}\pi(k\,|x_t)$$

$$=\sum_{k=2}^{n-1}\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{n-3}\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-3}{2})\sqrt{(n-3)\pi}(a_2-\frac{c_2^2}{a_1})^{-\frac{1}{2}}\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{\frac{1}{2}}}$$

$$\cdot\left[1+\frac{\left(\delta+\frac{\frac{c_2d_2}{a_1}-b_2}{a_2-\frac{c_2^2}{a_1}}\right)^2}{(n-3)\left(\sqrt{(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}})/(n-3)(a_2-\frac{c_2^2}{a_1})}\right)^2}\right]^{-\frac{n-3+1}{2}}\pi(k\,|x_t)$$

$$=\sum_{k=2}^{n-1}t\left(\delta;n-3,-\frac{\frac{c_2d_2}{a_1}-b_2}{a_2-\frac{c_2^2}{a_1}},\sqrt{(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}})/(n-3)(a_2-\frac{c_2^2}{a_1})}\right)\cdot\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{n-3}\pi(k\,|x_t).$$

Therefore, the Bayesian estimator of δ is

$$\hat{\delta}=\sum_{k=2}^{n-1}\frac{\frac{b_2-c_2d_2}{a_1}}{a_2-\frac{c_2^2}{a_1}}\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{n-3}\pi(k\,|x_t).$$

A.2.4. Proof of Bayesian estimator of ϕ_0

Finally, we consider

$$\pi(\phi_0\mid x_t)\propto\sum_{k=2}^{n-1}(M_1-\frac{(b_2-\phi_0c_2)^2}{a_2})^{-\frac{n-2}{2}}(a_1a_2-c_2^2)^{\frac{1}{2}}\left[e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}}a_2^{-\frac{1}{2}}\pi(k\,|x_t)$$

$$=\sum_{k=2}^{n-1}(a_1\phi_0^2-2d_2\phi_0+e_2-\frac{(b_2-\phi_0c_2)^2}{a_2})^{-\frac{n-2}{2}}(a_1a_2-c_2^2)^{\frac{1}{2}}\left[e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right]^{\frac{n-3}{2}}a_2^{-\frac{1}{2}}\pi(k\,|x_t)$$

$$=\sum_{k=2}^{n-1}t\left(\phi_0;n-3,-\frac{b_2c_2-a_2d_2}{a_1a_2-c_2^2},\sqrt{\frac{a_2e_2-b_2^2-\frac{(b_2c_2-a_2d_2)^2}{a_1a_2-c_2^2}}{(n-3)(a_1a_2-c_2^2)}}\right)a_2^{\frac{n-3}{2}}$$

$$\cdot\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{\frac{n-3}{2}}(a_2e_2-b_2^2-\frac{(b_2c_2-a_2d_2)^2}{a_1a_2-c_2^2})^{\frac{n-3}{2}}\pi(k\,|x_t).$$

Therefore, the Bayesian estimator of ϕ_0 is given by

$$\hat{\phi}_0=\sum_{k=2}^{n-1}\frac{a_2d_2-b_2c_2}{a_1a_2-c_2^2}\left(e_2-\frac{d_2^2}{a_1}-\frac{(\frac{c_2d_2}{a_1}-b_2)^2}{a_2-\frac{c_2^2}{a_1}}\right)^{\frac{n-3}{2}}(a_2e_2-b_2^2-\frac{(b_2c_2-a_2d_2)^2}{a_1a_2-c_2^2})^{\frac{n-3}{2}}a_2^{\frac{n-3}{2}}\pi(k\,|x_t).$$